THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2014-2015
Suggested Solution to Test 1

1. (a) Note $f'(z) = u_x + iv_x = 0$, so $u_x = v_x = 0$. By the Cauchy Riemann Equations, $v_y = u_x = 0$ and $v_x = -u_y = 0$. $u_x = u_y = v_x = v_y = 0$ for all $(x, y) \in \mathbb{R}^2$ and so u and v are constant functions. Therefore, $f(z)$ is a constant function.

   (b) Let $f(z) = \cos^2 z + \sin^2 z$. Then $f'(z) = -2\cos z \sin z + 2\sin z \cos z = 0$ for all $z \in \mathbb{C}$, and so $f(z)$ is a constant function. Note that $f(0) = 1$, so $f(z) \equiv 1$.

2. (a) Note that $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$ and $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$.

$$\begin{aligned}
\sin z + \cos z &= 2 \\
\frac{e^{iz} - e^{-iz}}{2i} + \frac{e^{iz} + e^{-iz}}{2} &= 2 \\
(1+i)(e^{iz})^2 - 4ie^{iz} + (i-1) &= 0 \\
e^{iz} &= \frac{2 \pm \sqrt{2}}{2} + i\frac{2 \pm \sqrt{2}}{2} \\
z &= \frac{\pi}{4} + 2k\pi - i\ln(\sqrt{2} \pm 1) \qquad \text{where } k \text{ is an integer.}
\end{aligned}$$

   (b) Let $z = x + iy$, then

$$\begin{aligned}
\mathrm{Log}(e^z) &= 2 \\
\mathrm{Log}(e^{x+iy}) &= 2 \\
\ln e^x + i\mathrm{Arg}(e^{x+iy}) &= 2 + i(0) \\
x + i\mathrm{Arg}(e^{x+iy}) &= 2 + i(0)
\end{aligned}$$

   Therefore, $x = 2$ and $y = 2k\pi$ where k is an integer.

3. Length of $C = 4\pi$ and $|e^z| = e^{\mathrm{Re}(z)} \le e^2$.
   z lies on C implies $z^2 + 1$ lies on the circle $\{|z - 1| = 4\}$, so

$$\begin{aligned}
|z^2 + 1| &\ge 3 \\
\frac{1}{|z^2 + 1|} &\le \frac{1}{3}
\end{aligned}$$

   By ML-estimate, $\left| \displaystyle\int_C \frac{e^z}{z^2 + 1}\, dz \right| \le \dfrac{4\pi e^2}{3}$.

4. (a) $f(z) = f(x + iy) = u(x, y) + iv(x, y) = \sqrt{|xy|} + i0$, so $u(x, y) = \sqrt{|xy|}$ and $v(x, y) = 0$.

$$\begin{aligned}
u_x(0, 0) &= \lim_{\delta x \to 0} \frac{u(\triangle x, 0) - u(0, 0)}{\triangle x} \\
&= \lim_{\triangle x \to 0} \frac{\sqrt{|\triangle x \cdot 0|}}{\triangle x} \\
&= 0 \\
&= v_y(0, 0)
\end{aligned}$$

and

$$\begin{aligned} u_y(0,0) &= \lim_{\triangle y\to 0}\frac{u(\triangle y,0)-u(0,0)}{\triangle y} \\ &= \lim_{\triangle y\to 0}\frac{\sqrt{|\triangle y\cdot 0|}}{\triangle y} \\ &= 0 \\ &= -v_x(0,0) \end{aligned}$$

Therefore, the Cauchy Riemann equations are satisfied at the point $z=0$

(b) If $f(z)$ is differentiable at $z=0$, then $f'(0)=u_x(0,0)+iv_x(0,0)=0$.
However, we consider $\triangle z = t+it$, where $t>0$, then

$$\begin{aligned} \lim_{\triangle z\to 0}\frac{f(0+\triangle z)-f(0)}{\triangle z} &= \lim_{t\to 0^+}\frac{\sqrt{|it^2|}}{t+it} \\ &= \frac{1}{1+i} \\ &\neq 0 \end{aligned}$$

Therefore, f is not differentiable at $z=0$.